

The Wilson-Polchinski exact renormalization group equation

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Abstract

The critical exponent η is not well accounted for in the Polchinski exact formulation of the renormalization group (RG). With a particular emphasis laid on the introduction of the critical exponent η , I re-establish (after Golner, hep-th/9801124) the explicit relation between the early Wilson exact RG equation, constructed with the incomplete integration as cutoff procedure, and the formulation with an arbitrary cutoff function proposed later on by Polchinski. I (re)-do the analysis of the Wilson-Polchinski equation expanded up to the next to leading order of the derivative expansion. I finally specify a criterion for choosing the “best” value of η to this order. This paper will help in using more systematically the exact RG equation in various studies.

Key words: Exact renormalization group, Derivative expansion, Critical exponents

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The renormalization group theory is specifically adapted to treat physical situations where infinitely many scales are (continuously) coupled [1] (on this

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point see [2] for example and a recent talk in [3]). In the study of such situations one may expect to have to do a complicated nonperturbative calculation. No doubt in the spirit of Wilson: his renormalization group formulation provided us with a nonperturbative tool to treat a nonperturbative problem [2]. However, in practice, the perturbative approach has appeared more attractive, surprisingly very efficient —so far as to produce very accurate estimates of (nonperturbative in nature) universal quantities such as critical exponents [4]— and exposed at length in any text book dealing with the renormalization group. On the contrary, a nonperturbative formulation of the renormalization group —proposed since 1970 [5] and named the “exact” renormalization group equation [6] (ERGE) to distinguish it from the discretized version— has little been studied before the ninety’s [7] due to its disheartening complexity (an integro-differential equation whose study requires numerical approximations or truncations not very well controlled). Even after fifteen years of resurgence of interest in the ERGE [7], there is, apparently, little success and when there are new interesting results [8], it seems difficult to establish their reliability because we are usually not familiar with nonperturbative approaches.

Contrary to the generally admitted idea [9, p. 588] I do not think that the calculations done with the ERGE are less precise than those obtained in using perturbative approaches. In fact one must compare what is effectively comparable. One easily forgets that the success of perturbation theory in estimating critical exponents of the Ising and n -vector models is based on a tremendous numerical tour de force which has provided us with seven orders of the perturbative series [10] combined with clever resummation techniques of divergent series, techniques themselves based on highly nontrivial estimations of large order behavior of these series [4,11]. In summary, this outstanding success is

exceptional and one surely not be able to reproduce it every day in any other domain of physical interest (however see [12]). It is possible that studies of the ERGE be never so developed to reach such a success. But if, more reasonably, one compares the results of the ERGE to those obtained with the most commonly used perturbation technique, namely the ϵ -expansion up to, say at most $O(\epsilon^2)$ then quantitatively the comparison is far from being ridiculous (see the end of the paper). At the qualitative level, one should even be able, sometimes in studying an ERGE, to show that the perturbative approach has failed (or will fail) in treating correctly some specific physical situation. Let me remind the reader with a remark relative to the use of the ERGE: ...[perhaps one day] “one will be able to develop approximate forms of the transformation which can be integrated numerically; if so one might be able to solve problems which cannot be solved any other way” [6, p. 153]. Actually, as I show in an other paper [13], the so-called Lifshitz tricritical point effectively involves a marginally relevant coupling which defies the perturbative framework.

With a view to consider correctly the ERGE adapted to the study of the Lifshitz point [13] (which involves two correlation lengths instead of one for the ordinary critical point), it is necessary to come back to the derivation of the ERGE on one aspect: the introduction of the critical exponent η . Actually a misleading account of this parameter exists since the famous paper by Wegner and Houghton [14] and has been repeated by the authors (including those of [7]) who have tried to determine (or to report on) the value of η from the study of the Polchinski ERGE [15,16]. On the contrary, the study of Golner [17] who considered the original Wilson version has been correctly done. As for studies using the ERGE satisfied by the Legendre transformed action (or the one-particle-irreducible vertex function) as in [18] or in [19] they are basically

correct although the introduction of η is effectuated by ad hoc dimensional arguments which, in my opinion, circumvent the crucial point of the many scales involved in the problem. The same kind of appreciation goes also for the calculations done along the line of the effective average action [20] initiated by Wetterich [21] (for a review see [22]) and for the calculations done with a proper time regulator [23]. In the following I will not consider further these ERGE's since the Wilson-Polchinski equation is much simpler and allows as well a correct investigation of the fixed point properties and of the associated critical exponents what is sufficient to illustrate my purposes.

Now, let me show the origin of the difficulty.

One currently presents the RG transformation of an action $S[\phi]$ as consisting of two steps:

- (1) an integration of the high momentum components of the field generating an effective action with a reduced cutoff.
- (2) a rescaling of the momenta back to the initial value of the cutoff accompanied by a re-normalization of the field $\phi \rightarrow \zeta \phi$ (ζ being related to η).

In this view the two steps are well separated (besides only the step 1 is considered in [24]) and it is a matter of fact that in considering the construction of the ERGE one never makes reference to any momentum scale other than the current scale Λ , except Golner [25]. In particular, one does not keep track of the history of the scale changes starting from some initial momentum scale Λ_0 ; the whole procedure is usually done “instantaneously” [26].

However, in doing so, one forgets two (closely related) Wilson's prescriptions:

- a) the re-normalization of the field (ζ) must be chosen in such a way as to keep one term of the action unchanged (*since the beginning*).
- b) the final RG equation must not depend explicitly on the renormalization “time” $\ell = \Lambda/\Lambda_0$.

Prescription a) clearly imposes a memory of the different changes of scale starting from some initial scale Λ_0 . This memory is usually not considered though it is inherent to the physics of many scales under study. Prescription b) is a consequence of a), it implies that the explicit scale dependence induced by a) be compensated by an appropriate choice of the cutoff function. Morris [18] satisfies this prescription when he considers a cutoff function $C(q, \Lambda)$ adapted to the scaling behavior expected at a fixed point but he writes: $C(q, \Lambda) = \Lambda^{\eta-2} \tilde{C}\left(\frac{q^2}{\Lambda^2}\right)$ without any reference to some Λ_0 which though would have been necessary from simple dimensional analysis arguments (Λ is dimensionfull and η is not an integer).

The expected relation of equivalence, simply noted by Morris [27], between the Wilson (with a specific cutoff procedure named the incomplete integration) and Polchinski (with an arbitrary cutoff function) versions of the ERGE has been explicitly established in a recent work by Golner [25]. It is, however, useful to re-consider explicitly the derivation of the Polchinski ERGE including the rescaling procedure.

In order to fully take into account the prescriptions a) and b) listed above, the starting point is a modified version of Polchinski’s action [24]. Consider an action of the following form:

$$S[\phi] = \frac{1}{2} \int_q \phi_q P^{-1}\left(\frac{q^2}{\Lambda^2}, \ell\right) \phi_{-q} + S_{\text{int}}[\phi] \quad (1)$$

in which $\ell = \frac{\Lambda}{\Lambda_0}$ is the current value of the (running) momentum cutoff Λ measured in some fixed initial momentum scale Λ_0 . $P(\frac{q^2}{\Lambda^2}, \ell)$ is some dimensionless cutoff function which depends explicitly on ℓ , contrary to the original Polchinski's cutoff function. This dependency makes a reference to some initial scale, it is inherent to the many scale problem that one is supposed to consider here. It is shown below that, near a fixed point of relevance for the study of an ordinary critical point, the explicit ℓ -dependence factorizes:

$$P(\frac{q^2}{\Lambda^2}, \ell) = \ell^{2\varpi} \tilde{P}(\frac{q^2}{\Lambda^2}) \quad (2)$$

with $\varpi = 1 - \frac{\eta}{2}$. Notice then that the factorized ℓ -dependence in (2) is identical to the $(\frac{\Lambda}{\Lambda_0})^{2-\eta}$ factor introduced by Golner [25] in his own cutoff function but from ad hoc dimensional analysis involving the anomalous dimension that the field acquires at a fixed point —see also Morris in [18]. In fact there is no need to call any anomalous dimensional analysis in, it suffices to follow the rules established by Wilson.

The Polchinski flow equation, which corresponds to the step 1 described above (associated to the partial transformation $\mathcal{G}_{\text{tra}}S[\phi]$ and which reduces infinitesimally the range of integration of q) is unchanged in its general form [7] and reads ($\dot{S} = -\Lambda \frac{dS}{d\Lambda} = \mathcal{G}_{\text{tra}}S + \mathcal{G}_{\text{dil}}S$):

$$\mathcal{G}_{\text{tra}}S[\phi] = \frac{1}{2} \int_q \Lambda \frac{\partial P}{\partial \Lambda} \left[\frac{\delta^2 S}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S}{\delta \phi_q} \frac{\delta S}{\delta \phi_{-q}} + 2P^{-1} \phi_q \frac{\delta S}{\delta \phi_q} \right] \quad (3)$$

in which expression the field variable ϕ_q is such that $0 < |\mathbf{q}|/\Lambda < 1 - d\ell/\ell$ (i.e. the rescaling of the above step 2 has not yet been considered). Now occurs the re-normalization of the field at fixed momentum scale:

$$\phi_q = \zeta (\ell - d\ell) \phi_q'' \quad (4)$$

This step is necessary because one must define a new field so as to keep the original physics unchanged (this is equivalent to the Kadanoff prescription [28]). Notice here the implicit but obliged reference to an initial scale of reference Λ_0 (associated to some original physics) via ℓ . Below it is shown that near a fixed point of relevance in the study of an ordinary critical point, $\zeta(\ell)$ must have the following form:

$$\zeta(\ell) = \ell^\varpi \quad (5)$$

$\zeta(\ell)$ induces an explicit dependence on ℓ in (3) once expressed in terms of the field ϕ_q'' :

$$\mathcal{G}_{\text{tra}}S[\phi] = \frac{1}{2} \int_q \Lambda \frac{\partial P}{\partial \Lambda} \left[\ell^{-2\varpi} \left(\frac{\delta S}{\delta \phi_q''} \frac{\delta S}{\delta \phi_{-q}''} - \frac{\delta^2 S}{\delta \phi_q'' \delta \phi_{-q}''} \right) - 2P^{-1} \phi_q'' \frac{\delta S}{\delta \phi_q''} \right] \quad (6)$$

Notice [29] that since (3) is already of order $d\ell$ one may neglect the correction proportional to $d\ell$ in (4).

Requiring that the final equation must not depend explicitly on ℓ (see [6, p. 126]), then the anticipated ℓ -dependence of the cutoff function P is linked to that of ζ and using (5) one obtains (2).

Finally the rescaling is performed and gives $\mathcal{G}_{\text{dil}}S$ which must globally account for the transformation of S under the change $\phi_q \rightarrow \phi_{q'}'$ where q' is rescaled back as $q' = (1 + \frac{d\ell}{\ell})q$ (here I consider dimensionless q). This means that $\mathcal{G}_{\text{dil}}S$ includes also the re-normalization of the field given in (4).

Consequently there are two sources of re-normalisation of the field in $\mathcal{G}_{\text{dil}}S$: one coming from $\zeta(\ell - d\ell)$ and the other from the actual rescaling to the

original momentum scale which expresses as follows:

$$\phi_q'' = s^y \phi'_{sq=q'} \quad (7)$$

wheres $= 1 + \frac{d\ell}{\ell}$ and y is simply determined by usual dimensional analysis and has already been implicitly fixed by choosing a dimensionless cutoff function P in (1). This yields $y = \frac{d}{2}$ and thus $\phi_q = \left(1 - \frac{d\ell}{\ell}\right)^\varpi \left(1 + \frac{d\ell}{\ell}\right)^{\frac{d}{2}} \ell^\varpi \phi'_{q'}$ which gives:

$$\mathcal{G}_{\text{dil}} S = \int_q \left[\left(\frac{d}{2} - \varpi \right) \phi_q + \mathbf{q} \cdot \partial_q \phi_q \right] \frac{\delta S}{\delta \phi_q} \quad (8)$$

in which $\mathbf{q} \cdot \partial_q = \sum_{\mu=1}^d q_\mu \frac{\partial}{\partial q_\mu}$

By adding up (6) and (8), the complete Polchinski equation finally reads (as already said above, the explicit Λ -dependence in $\tilde{P}(\frac{q^2}{\Lambda^2})$ has been absorbed in a dimensionless q):

$$\begin{aligned} \dot{S} = & \int_q \left[\left(\frac{d}{2} - \varpi \right) \phi_q + \mathbf{q} \cdot \partial_q \phi_q \right] \frac{\delta S}{\delta \phi_q} \\ & + \int_q \left[\varpi \tilde{P}(q^2) - q^2 \tilde{P}'(q^2) \right] \left(\frac{\delta^2 S}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S}{\delta \phi_q} \frac{\delta S}{\delta \phi_{-q}} + 2\tilde{P}^{-1} \phi_q \frac{\delta S}{\delta \phi_q} \right) \quad (9) \end{aligned}$$

in which $\tilde{P}(x)$ is defined in (2) and $\tilde{P}'(x) = \frac{d\tilde{P}(x)}{dx}$.

Compared to the Wilson equation:

$$\begin{aligned} \dot{S} = & \int_q \left[\left(\frac{d}{2} \phi_q + \mathbf{q} \cdot \partial_q \phi_q \right) \right] \frac{\delta S}{\delta \phi_q} \\ & + \int_q \left(c + 2q^2 \right) \left(\frac{\delta^2 S}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S}{\delta \phi_q} \frac{\delta S}{\delta \phi_{-q}} + \phi_q \frac{\delta S}{\delta \phi_q} \right) \quad (10) \end{aligned}$$

it is easy to show that, provided $\varpi = c$, one obtains [25] Wilson from Polchinski by choosing $\tilde{P}(q^2) = e^{-2q^2}$ and by making the redundant change $\phi_q \rightarrow e^{-q^2} \phi_q$.

The relation of ϖ to the critical exponent η is obtained from eqs (4,5) which induce a redefinition of the field with a view to keep the physics unchanged compared to the physics described by the original field. Notice that this renormalization of ϕ occurs before the rescaling of the momenta has been performed. We are in position to use the trivial changes of [30, p. 592] under the exact form of equations 2.22 and 2.23 (see also [31]). Let me consider the two-point correlation function in terms of the original field:

$$\langle \phi_p \phi_{-p} \rangle_S = G(p, S) \quad (11)$$

Provided one is only interested to momenta $q = \ell p$ (with $\ell < 1$) then one has [using (4) with $\ell - d\ell \rightarrow \ell$]:

$$\phi_q = \zeta(\ell) \phi_q'' \quad (12)$$

$$\langle \phi_p \phi_{-p} \rangle_S = [\zeta(\ell)]^2 \langle \phi_q'' \phi_{-q}'' \rangle_{S'} \quad (13)$$

where $S = S[\phi]$ and $S' = S'[\phi'']$. Finally it comes:

$$G(p, S) = [\zeta(\ell)]^2 G(\ell p, S') \quad (14)$$

which is a strong constraint on G . Indeed, following Fisher [31], one may easily show that at a fixed point S^* , $G^*(p)$ and $\zeta(\ell)$ have the respective following forms:

$$G^*(p) = \frac{D}{p^\vartheta} \quad (15)$$

$$\zeta(\ell) = \ell^{\frac{\vartheta}{2}} \quad (16)$$

with $\vartheta = 2\zeta'(1)$ and $\zeta'(\ell) = \frac{d\zeta}{d\ell}$. Consequently (15,16) give the relation $\varpi = \vartheta/2$ and finally since (15) is precisely the scaling behavior expected for the two

point correlation function at an ordinary critical point with $\vartheta = 2 - \eta$, then one obtains the relation we were looking for:

$$\varpi = 1 - \frac{\eta}{2} \quad (17)$$

This ends the derivation of the Wilson-Polchinski ERGE.

Since the previous studies of the Polchinski equation up to $O(\partial^2)$ in the derivative expansion [15,16] have not been done correctly [the term proportional to $\varpi \tilde{P}(q^2)$ in (9) was not considered], I find it useful (despite Golner's study [17]) to reconsider it at the light of the established equivalence between the Wilson and Polchinski versions.

I consider eq. (9) expanded up to the second derivative of the field. This means that the action is limited to the form:

$$S[\phi] = \int dx \left[Z(\phi) (\partial\phi)^2 + V(\phi) \right] \quad (18)$$

in which $V(\phi)$ and $Z(\phi)$ are two arbitrary fonctionnals of $\phi(x)$. Before writing down the equations satisfied by $V(\phi)$ and $Z(\phi)$ as consequence of eq. (9) truncated to actions of type (18), let me introduce two modifications. First, in order to allow a comparison with the work of Golner [17], I considers a redundant transformation of the field:

$$\tilde{\phi}_q = \psi(q^2) \phi_q$$

in which $\psi(q^2)$ is arbitrary except the normalization $\psi(0) = 1$. Secondly, for practical reason, I subtract the high temperature fixed point $\frac{1}{2} \int_p \left(\tilde{P} \psi^2 \right)^{-1} \phi_p \phi_{-p}$ from the action. So that in terms of the new action and the new field (again denoted respectively S and ϕ), eq. (9) reads:

$$\begin{aligned}\dot{S} = & - \int_q \phi_q \left(\tilde{d}_\phi + 2q^2 \frac{\psi'}{\psi} + \mathbf{q} \cdot \partial_q \right) \frac{\delta S}{\delta \phi_q} \\ & + \int_q \left(\varpi \tilde{P} - q^2 \tilde{P}' \right) \psi^2 \left[\frac{\delta^2 S}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S}{\delta \phi_q} \frac{\delta S}{\delta \phi_{-q}} \right]\end{aligned}\quad (19)$$

in which $\tilde{d}_\phi = \frac{d}{2} + \varpi$ and $\psi' = d\psi/dq^2$.

Two other useful modifications are introduced in the derivative expansion. First I rescale the field $\phi = I_0^{1/2} \bar{\phi}$ and the potential $V = I_0 \bar{V}$ where $I_0 = \int_q \left(\varpi \tilde{P} - q^2 \tilde{P}' \right) \psi^2$ and instead of V , I considers the equation for $v_1 = d\bar{V}/d\bar{\phi}$. Finally, for the sake of unified notations I rename Z as v_2 . With these new definitions, the RG equations $O(\partial^2)$ read:

$$\dot{v}_1 = v_1'' + d_\phi v_1 - \left(\tilde{d}_\phi \bar{\phi} + 2\varpi v_1 \right) v_1' + P_1 v_2' \quad (20)$$

$$\dot{v}_2 = v_2'' - 2(\varpi + 1) v_2 - \left(\tilde{d}_\phi \bar{\phi} + 2\varpi v_1 \right) v_2' + v_1' (P_2 v_1' - 2\psi_0' - 4\varpi v_2) \quad (21)$$

in which the prime means the derivative with respect to $\bar{\phi}$ and $d_\phi = \frac{d}{2} - \varpi$, $\psi_0' = d\psi/dq^2|_{q=0}$, $P_1 = 2\frac{I_1}{I_0}$ with $I_1 = \int_q q^2 \left(\varpi \tilde{P} - q^2 \tilde{P}' \right) \psi^2$ and $P_2 = - \left[\tilde{P}'_0 (\varpi - 1) + 2\varpi \psi_0' \right]$ with $\tilde{P}'_0 = d\tilde{P}/dq^2|_{q=0}$. In order to numerically study this set of second order differential equations I make the following choice:

$$\tilde{P}(q^2) = e^{-aq^2} \quad (22)$$

$$\psi(q^2) = \frac{1}{1 + bq^2} \quad (23)$$

with a and b two parameters on which the value of the critical exponents do not theoretically depend according to the so-called (renormalization) scheme independence (a) and reparametrization invariance (b). But since these two general properties are no longer satisfied in the derivative expansion, I shall obtain values of the critical exponents that depend on a and b . The main question will then be to determine the values of these two parameters which give the best value of η to the order considered.

The study of eqs (20,21) is by now standard [32]. The fixed point equations $\dot{v}_1 = \dot{v}_2 = 0$ are two coupled differential equations of second order for two ordinary functions of the real variable $\bar{\phi}$. These equations may be numerically integrated using the shooting method with the Newton-Raphson algorithm (see, for example, [33]): starting from a sufficiently large value $\bar{\phi}_0$ of $\bar{\phi}$ where the regular large $\bar{\phi}$ behavior of the v_i 's is imposed, one shoots towards the origin $\bar{\phi} = 0$ where one checks for even (if one is looking for a Z_2 -symmetric fixed point, odd otherwise) functions V and Z .

The large $\bar{\phi}$ behavior of the solutions corresponding to eqs (20,21) looks like [34]:

$$v_{1\text{asy}} = G_1 \bar{\phi}^{\theta_1} + \theta_1 G_1^2 \bar{\phi}^{2\theta_1-1} + \dots \quad (24)$$

$$v_{2\text{asy}} = b\theta_1 G_1 \bar{\phi}^{\theta_1-1} + \dots + G_2 \bar{\phi}^{\theta_2} + \dots \quad (25)$$

with $\theta_1 = \frac{d-2\varpi}{d+2\varpi}$ and $\theta_2 = -4\frac{1+\varpi}{d+2\varpi}$; G_1 and G_2 must be adjusted at $\bar{\phi} = \bar{\phi}_0$ in order to reach the origin with the following conditions satisfied (for a Z_2 -symmetric fixed point):

$$v_1(0) = 0 \quad (26)$$

$$v_2'(0) = 0 \quad (27)$$

Since eqs (20,21) are of second order the general solution involves four arbitrary constants which are fixed by G_1 , G_2 and the two conditions (26,27). As for the determination of $\eta = 2(1 - \varpi)$, it is associated with the additional condition:

$$v_2(0) = Z_0 \quad (28)$$

in which Z_0 is some arbitrarily fixed constant (the value of $Z(0)$). In principle,

η should not vary with Z_0 , but since the reparametrization invariance is broken by the derivative expansion, one will obtain a non trivial function $\eta(Z_0)$ which, however, should show, as vestige of the invariance, an extremum to which is associated a zero eigenvalue [35] (see below). The value of η at this extremum (η^{opt}) being its optimized best value at the order of the derivative expansion considered (for a and b fixed).

As usual, I find only one fixed point satisfying the Z_2 -symmetry, it is characterized by a value of G_1 which is close to that obtained at the leading order of the derivative expansion: $G_1 \simeq -2.3$ (which now depends on the value of Z_0). Below, I explain how I have practically chosen the optimum value of Z_0 .

Once the fixed point has been determined (depending on a and b), one looks at the eigenvalue equations obtained by linearizing eqs (20,21) about a fixed point solution v_i^* . By setting $v_i = v_i^* + \varepsilon e^{\lambda t} g_i$ (with $t = -\ln \ell$) and retaining the linear term in ε , the eigenvalue equations read:

$$g_1'' = (\lambda - d_\phi + 2\varpi v_1^{*'}) g_1 + (\tilde{d}_\phi \bar{\phi} + 2\varpi v_1^*) g_1' - P_1 g_2' \quad (29)$$

$$g_2'' = [\lambda + 2(\varpi + 1 + 2\varpi v_1^{*'})] g_2 + (\tilde{d}_\phi \bar{\phi} + 2\varpi v_1^*) g_2' + 2\varpi v_2^{*'} g_1 + 2(2\varpi v_2^* - P_2 v_1^{*'} - b) g_1' \quad (30)$$

the interesting solutions of which are again looked for by the shooting method such that they satisfy the following large $\bar{\phi}$ behavior at some initial value $\bar{\phi}_0$ (with $\varkappa_1 = \frac{d-2\varpi-2\lambda}{d+2\varpi}$, $\varkappa_2 = -2\frac{2+2\varpi+\lambda}{d+2\varpi}$):

$$g_{1asy} = S_1 \bar{\phi}^{\varkappa_1} + \dots \quad (31)$$

$$g_{2asy} = b S_1 \varkappa_1 \bar{\phi}^{\varkappa_1-1} + S_2 \bar{\phi}^{\varkappa_2} + \dots \quad (32)$$

and the symmetric property at the origin $\bar{\phi} = 0$:

$$g_1(0) = 0 \quad (33)$$

$$g_2'(0) = 0 \tag{34}$$

One could think that the four arbitrary constants of integration are all fixed by the determination of S_1 , S_2 and the two conditions (33,34) leaving no room to any specific determination of the eigenvalue λ . In fact, as in any eigenvalue problem, the global normalization of the eigenvectors may be chosen at will. For example, one may arbitrarily fix $S_1 = 1$ and this allows the determination of quantized values for λ [16].

With the ERGE, one may also easily considers the asymmetric case (corresponding to an action S involving odd powers of ϕ). To this end it is sufficient to modify eqs (33,34) into:

$$g_1'(0) = 0 \tag{35}$$

$$g_2(0) = 0 \tag{36}$$

As mentioned above, the optimized value η^{opt} associated with a fixed point that satisfies the reparametrization invariance is associated to a zero eigenvalue [35]. I have used this property to determine η^{opt} directly from a set of differential equations instead of determining it by a numerical estimation of a derivative with respect to Z_0 . The procedure consists in considering the two fixed point equations (20,21) with $\dot{v}_1 = \dot{v}_2 = 0$ together with the two eigenvalue equations (29,30) with λ fixed to zero while the condition (28) is removed.

As in the Golner study [17], I observe a strong dependence (essentially linear) of η^{opt} on the parameter b (of the redundant function ψ). This had led Golner to propose a criterion of choice based on a global minimization of the function $Z(\phi)$ (presently corresponding to v_2). With regard to the few order of the

derivative expansion considered here, there is no unquestionable criterion of choice, simply the Golner criterion seems reasonable [40]. However it appears that with the second parameter a (of the cutoff function) one can make $Z(\phi)$ globally as small as one wants for any value of b (and thus for essentially any value of η^{opt}). Fortunately, it appears also that at fixed value of η^{opt} (fixed value of b) the geometrical form of the function $Z(\phi)$ is independent of a except in magnitude (there is some similitude transformation which links the different curves $Z(\phi)$ for a fixed b). Hence it suffices to choose a specific form of $Z(\phi)$ to get a unique value of η^{opt} (at a given b but independent of a). I have arbitrarily chosen a family of $Z(\phi)$ such that the first two extrema of the function (starting from the origin) are disposed symmetrically with respect to the ϕ -axis. This yields, approximately, the value:

$$\eta^{\text{opt}} \simeq 0.025 \tag{37}$$

From this choice of η the determination of the eigenvalues follows without ambiguity.

In the symmetric case there is a unique positive eigenvalue λ_1 related to the critical exponent ν ($\nu = 1/\lambda_1$), I obtain $\nu \simeq 0.60$; the following eigenvalue λ_2 is negative and is related to the correction exponent ω ($\omega = -\lambda_2$), I obtain $\omega \simeq 0.87$. Of course there is also the zero eigenvalue corresponding to the definition of η^{opt} . If instead of V' I had considered the potential V I would have obtained the supplementary trivial eigenvalue $\lambda_0 = d$.

In the asymmetric case I find two positive eigenvalues $\check{\lambda}_1$ and $\check{\lambda}_2$ corresponding respectively to $(d + 2 - \eta^{\text{opt}})/2$ (associated with the magnetic-like linear coupling $h\phi$) and to $(d - 2 + \eta^{\text{opt}})/2$ (associated with a redundant ϕ^3 -like term in

S). The first interesting eigenvalue is the negative one $\check{\lambda}_3$ related to correction exponent ω_5 ($\omega_5 = -\check{\lambda}_3$) that controls the leading correction-to-scaling term due to the asymmetry, I obtain $\omega_5 \simeq 1.49$.

The same analysis may be done at the lowest order of the derivative expansion (local potential approximation), in that case the study is independent of a and b and one finds (for $\omega_{5\text{LPA}}$ see also [7,36]): $\eta_{\text{LPA}} = 0$, $\nu_{\text{LPA}} = 0.64956$, $\omega_{\text{LPA}} = 0.65574$, $\omega_{5\text{LPA}} = 1.8867$.

Of course, as indicated in the introduction, the quality of these results cannot be compared with that of the best estimates obtained from perturbation theory [37] for a review see [4]: $\nu = 0.6304 \pm 0.0013$, $\eta = 0.0335 \pm 0.0025$, $\omega = 0.799 \pm 0.011$ (although the present estimates are not so bad regarding the low order of the expansion). In fact, a more decent comparison is with the ϵ -expansion up to $O(\epsilon^2)$ (or even up to $O(\epsilon^3)$). One knows that this latter expansion is an asymptotic series the first two or three terms of which only seem to converge. For example by setting $\epsilon = 1$ at orders 1, 2 and 3 for η one obtains respectively 0., 0.0185, 0.0372. The value displayed in (37) compares favourably with these numbers. The question is now to know the effect of the next order in the derivative expansion. It is presently under study [38].

The case of ω_5 plainly illustrates my purpose because the best perturbative estimate has only been done up to $O(\epsilon^3)$ [39] and it is almost impossible to get a true estimate from the series as shown by the sequence: 2.83, 0.72, 7.36 (the near diagonal Padé approximants yield instead the sequence: 2.83, 1.85, 2.32) compared to 1.89, 1.49 in the present study, results obtained very easily by simply changing the conditions of integration of the equations. One sees that even if no error estimate can still be given, the derivative expansion up

to $O(\partial^2)$ compares favourably with the ϵ -expansion up to $O(\epsilon^2)$, even up to $O(\epsilon^3)$.

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References

- [1] K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975).
- [2] K. G. Wilson, in Phase Transitions and Critical Phenomena Vol. **VI**, p. 1, *Ed. by* C. Domb and M.S. Green (Acad. Press, N.-Y., 1976).
- [3] J. Zinn-Justin, in Progress in Mathematical Physics Vol. **30**, p. 55, *Ed. by* B. Duplantier and V. Rivasseau (Birkhauser-Boston, 2002).
- [4] J. Zinn-Justin, Phys. Rep. **344**, 159 (2001).
- [5] K. G. Wilson, in “*Irvine Conference (unpublished)*” (1970).
- [6] K. G. Wilson and J. Kogut, Phys. Rep. **12C**, 77 (1974).
- [7] C. Bagnuls and C. Bervillier, Phys. Rep. **348**, 91 (2001).
- [8] M. Tissier, B. Delamotte and D. Mouhanna, Phys. Rev. Lett. **84**, 5208 (2000);
O. Lauscher and M. Reuter, Phys. Rev. **D65**, 025013 (2001).
- [9] A. Pelissetto and E. Vicari, Phys. Rep. **368**, 549 (2002).
- [10] B. G. Nickel, D. I. Meiron and G. A. Baker, Jr, “*Compilation of 2-pt and 4-pt graphs for continuous spin models*”, Guelph University preprint, unpublished (1977).
- [11] F. Jasch and H. Kleinert, J. Math. Phys. **42**, 52 (2001).

- [12] H. Kleinert and V. I. Yukalov, cond-mat/0402163.
- [13] C. Bervillier, hep-th/0405027.
- [14] F. J. Wegner and A. Houghton, Phys. Rev. **A8**, 401 (1973).
- [15] R. D. Ball, P. E. Haagensen, J. I. Latorre and E. Moreno, Phys. Lett. **B347**, 80 (1995).
- [16] J. Comellas, Nucl. Phys. **B509**, 662 (1998).
- [17] G. R. Golner, Phys. Rev. **B33**, 7863 (1986).
- [18] T. R. Morris, Phys. Lett. **B329**, 241 (1994).
- [19] T. R. Morris and M. D. Turner, Nucl. Phys. **B509**, 637 (1998).
- [20] N. Tetradis and C. Wetterich, Nucl. Phys. **B422**, 541 (1994). L. Canet, B. Delamotte, D. Mouhanna and J. Vidal, Phys. Rev. **D67**, 065004 (2003); *ibid.***B68**, 064421 (2003). H. Ballhausen, hep-th/0303070.
- [21] C. Wetterich, Nucl. Phys. **B352**, 529 (1991).
- [22] J. Berges, N. Tetradis and C. Wetterich, Phys. Rep. **363**, 223 (2002).
- [23] A. Bonanno and D. Zappalà, Phys. Lett. **B504**, 181 (2001). M. Mazza and D. Zappalà, Phys. Rev. **D64**, 105013 (2001).
- [24] J. Polchinski, Nucl. Phys. **B231**, 269 (1984).
- [25] G. R. Golner, hep-th/9801124.
- [26] When the one-particle-irreducible vertex function is considered, two momentum scales of reference coexist for purely technical reasons: the current cutoff Λ is said to be an “infrared cutoff” and the customary “ultraviolet cutoff” Λ_0 is still mentioned but plays actually no role. With regards to the considerations of the present paper, the reader could be deceived by this apparent memory of the scale changes.

- [27] T. R. Morris, Phys. Lett. **B334**, 355 (1994).
- [28] L. P. Kadanoff, Physics **2**, 263 (1966).
- [29] Notice also that in [14] $\zeta(\ell)$ which is written $\zeta(1 - dt)$ is not considered as contributing to $\mathcal{G}_{\text{tra}}S[\phi]$ because precisely it would induce a term of higher order in dt . This argument could have been correct if one had not the prescription a) which imposes to keep exactly constant one term of the action. In addition, with a hard cutoff, prescription b) would have been difficult to realize.
- [30] S. K. Ma, Rev. Mod. Phys. **45**, 589 (1973).
- [31] M. E. Fisher, in “*Critical Phenomena*”, Lecture Notes in Physics, p. 1, *Ed. by* F.J.W. Hahne (Springer-Verlag, Pub., 1983).
- [32] T. R. Morris, Phys. Rev. Lett. **77**, 1658 (1996).
- [33] W. H. Press, B. P. Flannery, S. A. Teukolsky and W. T. Vetterling, in “*Numerical Recipes.*”, “*The Art of Scientific Computing*” (Cambridge University Press, 1986).
- [34] More terms are needed to numerically obtain a solution with some accuracy.
- [35] T. L. Bell and K. G. Wilson, Phys. Rev. **B10**, 3935 (1974).
- [36] M. M. Tsy-pin, Nucl. Phys. **B636**, 601 (2002). D. F. Litim and L. Vergara, Phys. Lett. **B581**, 263 (2004).
- [37] R. Guida and J. Zinn-Justin, J. Phys. **A31**, 8103 (1998).
- [38] C. Bagnuls, C. Bervillier and M. Shpot, in progress
- [39] F. C. Zhang and R. K. P. Zia, J. Phys. **A15**, 3303 (1982).
- [40] Criterion such as elaborated by D. F. Litim, Phys. Lett. **B486**, 92 (2000), being specific to the Legendre transformed ERGE, seems not applicable here.